Chapter 8: Functions

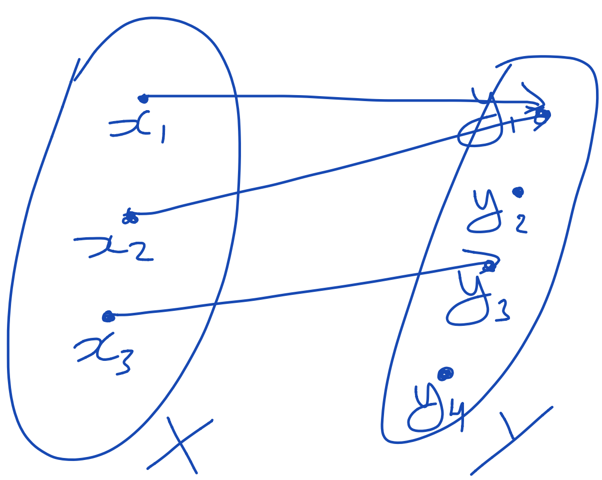
Defn: Let X and Y be two sets. A function, map, or mapping from X to Y is the assignment of a unique element of Y to each element of X.

We write
$$\boxed{\begin{aligned} f: X &\to Y \\ x &\mapsto \underbrace{f(x)}_{\text{value of } f \text{ at } x \,/\, \text{image of } x \text{ under } f} \end{aligned}}$$

to mean that the element $y = f(x) \in Y$ is assigned to $x \in X$.

We call X the domain of f, and we call Y its co-domain.

Pictorial representation



Here
$f(x_1) = y_1$
$f(x_2) = y_1$
$f(x_3) = y_3$.

<u>Note</u>: Every x in the domain X has an image $f(x)$. On the other hand, elements in the co-domain can be the image of one, or more elements of the domain. They can also <u>not</u> be the image of any x.

Example: There are 8 functions from $X = \{a, b, c\}$ to $Y = \{d, e\}$.

x	a	b	c
$f_1(x)$	d	d	d
$f_2(x)$	d	d	e
$f_3(x)$	d	e	d
$f_4(x)$	d	e	e
|	e	d	d
|	e	d	e
|	e	e	d
$f_8(x)$	e	e	e

So here, $f_1(a) = f_1(b) = f_1(c) = d$

while $f_4(a) = d$, $f_4(b) = f_4(c) = e$.

Above, the functions had different domains and co-domains

That need not be the case.

Example: Let $f: \mathbb{R} \to \mathbb{R}$
$$x \mapsto 2x$$

Example: Let $A = \{a, b\}$, and

$$g: A \to A.$$

In fact, we can have exactly 4 functions from A to A.

Exercise: List these 4 functions.

Def'n: Given a set X, the identity function

$$I_X : X \to X$$

is given by $I_X(x) = x \quad \forall x \in X.$

Important comment: The domain and co-domain of a function are *needed* to define it.

Functions with different domain or co-domain are different functions even if they have the same 'formula'.

Example: Let
$$f_1 : \mathbb{R} \to \mathbb{R}$$
$$x \to x^2,$$

$$f_2 : \mathbb{R}^{\geq} \to \mathbb{R}$$
$$x \to x^2,$$

$$f_3 : \mathbb{R} \to \mathbb{R}^{\geq}$$
$$x \to x^2,$$

$$f_4 : \mathbb{R}^{\geq} \to \mathbb{R}^{\geq}$$
$$x \to x^2.$$

These 4 functions are different.

Examples: Often, when we are working with functions on the real numbers, we omit specifying the domain, and co-domain.

Consider $f_1(x) = \dfrac{x^2}{x-1}$.

Here, we have to exclude $x=1$, from the domain (and think of $f_1 : \mathbb{R}\setminus\{1\} \to \mathbb{R}$) or we have to extend it, by doing something like

$$f_2(x) = \begin{cases} \frac{x^2}{x-1}, & x \neq 1 \\ 3\pi, & x = 1. \end{cases}$$

Example (The modulus function)

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto |x|$$

is defined via $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$

Equality of Functions: The functions $f: X \to Y$ and $g: X \to Y$ are equal when $f(x) = g(x)$ $\forall x \in X$.

Note: f & g have the same domain and codomain.

Restrictions of Functions: Let $f: X \to Y$ and let $A \subseteq X$. We can define $g: A \to Y$ via $\underline{g(x) = f(x) \ \forall x \in A}$. Such a function is called the restriction of f to A, and denoted $f|_A$.

Compositions of functions:

Let $f: X \to Y$ and $g: Y \to Z$, we can define a function $h: X \to Z$ via $h(x) = g(f(x))$ $\forall x \in X$. h is called the composition (or composite) of f and g and denoted $g \circ f : X \to Z$, with

$$(g \circ f)(x) = g(f(x)) \quad \forall x \in X.$$

$$g \circ f : X \xrightarrow{f} Y \xrightarrow{g} Z$$

Examples: The function $f: \mathbb{R} \to \mathbb{R}$, $x \to (2x)^2$ satisfies $f = g \circ h$ where $g: \mathbb{R} \to \mathbb{R}$, $x \to x^2$

and $h: \mathbb{R} \to \mathbb{R}$, $x \to 2x$.

Proof: $(g \circ h)(x) = g(h(x)) = g(2x) = (2x)^2$
$= f(x) \ \forall x \in \mathbb{R}.$

Comment: Notice that $(h \circ g)(x) = 2x^2$, so that $h \circ g \neq g \circ h$.

Proposition: Let $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ be functions. Then

i) $(h \circ g) \circ f = h \circ (g \circ f) : X \to Z$ (associativity)

ii) $f \circ I_X = f = I_Y \circ f : X \to Y$.

Proof: (i) $((h \circ g) \circ f)(x) = (h \circ g)(f(x))$

$= h(g(f(x)) \ \forall x \in X.$

Similarly $(h \circ (g \circ f))(x) = h((g \circ f)(x))$

$= h(g(f(x))) \ \forall x \in X.$

Therefore $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) Exercise.

Sequences

A function $f : \mathbb{Z}^+ \to A$ is called a sequence (in the set A).

Examples:

- $f : \mathbb{Z}^+ \to \mathbb{Z}^+$
  $n \mapsto n^2$

- $f : \mathbb{Z}^+ \to \mathbb{R}$
  $n \longmapsto \dfrac{n}{n+1}$

- The Fibonacci sequence

- $f : \mathbb{Z}^+ \to \mathbb{R}$
  $n \longmapsto \sum_{i=1}^{n} i$

Definition: Let $f : \mathbb{Z}^+ \to \mathbb{R}$ be a sequence. We say the sequence is null, written $\lim_{n\to\infty} f(n) = 0$, when

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ \text{ s.t. } \forall n \in \mathbb{Z}^+ \ (n \geq N \Rightarrow |f(n)| < \varepsilon)$$

One way to interpret this: For any choice of the real number ε, you can find an integer N (that can depend on ε) so that $|F(N)| < \varepsilon$, $|F(N+1)| < \varepsilon$, $|F(N+2)| < \varepsilon$ ....

Since this is true for any ε (as small as we like), we can think of the sequence as converging to 0.

Example: Prove that $\lim_{n \to \infty} \frac{1}{n} = 0$.

Proof: We would like to prove that

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ \text{ s.t. } \forall n \in \mathbb{Z}^+ \left(n \geq N \Rightarrow \left|\frac{1}{n}\right| < \varepsilon \right).$$

To that end, consider any $\varepsilon > 0$, and notice that we can choose N to be a positive integer with $N > \frac{1}{\varepsilon}$, because $N > \frac{1}{\varepsilon} \Rightarrow \frac{1}{N} < \varepsilon$. Moreover, $\forall n \geq N$, $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$, so we are done. ∎

The image of a function:

Def'n: Let $f: X \to Y$ be a function. The image of f, denoted $Im(f)$ is defined by $Im(f) := \{f(x) \mid x \in X\}$.

Thus, $Im(f) \subseteq Y$, and consists of those elements of Y "that can be reached from X".

Thus $\left[\left(y \in Im(f)\right) \Leftrightarrow \left(\exists x \in X, f(x) = y\right)\right]$.

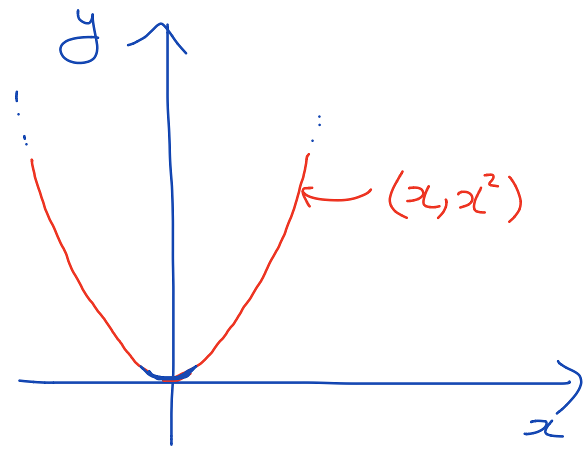
The graph of a function:

Def'n: Let $f: X \to Y$ be a function. The graph of f, denoted by G_f is defined by

$$G_f := \{(x, y) \in X \times Y \mid y = f(x)\}.$$

Example: $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$

$G_f := \{(x, x^2) \mid x \in \mathbb{R}\}$



Types of functions: Injections, surjections, and bijections

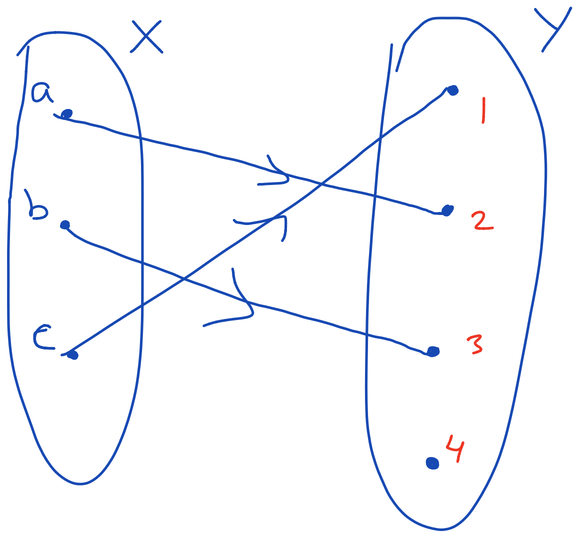
Let $f: X \to Y$ be a function. We say that f is an injection (or a one-to-one function) if

$$\boxed{\forall x_1, x_2 \in X,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)}$$

or equivalently, if

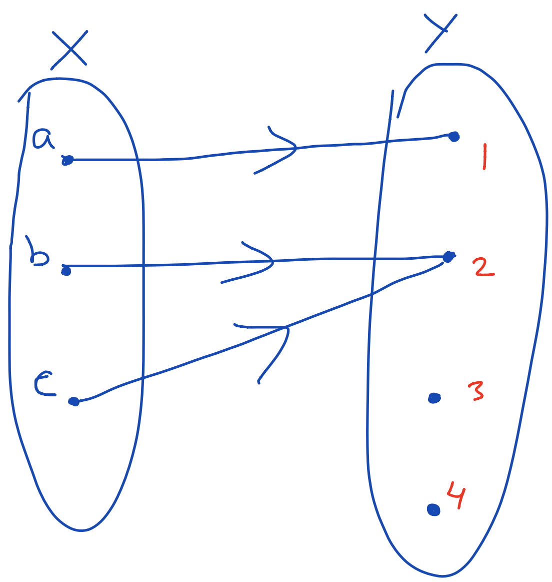
$$\boxed{\forall x_1, x_2 \in X,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2.}$$

Examples: • $f: X \to Y$ given by



is an injection.

On the other hand:



is not an injection because $f(b) = f(c)$ and $b \neq c$.

Let $f: X \to Y$ be a function. We say that f is ~~an~~ a surjection (or an onto function) if

$$\forall y \in Y, \exists x \in X \text{ s.t. } f(x) = y$$

In a sense, this says that we can 'reach' all elements of Y.

The two examples above did NOT depict surjective functions because in each case there was no x, such that $f(x) = 4$.

If $f: X \to Y$ is both injective and surjective, we say it is a bijection.

Definition: Given a function $f: X \to Y$, a pre-image of $y \in Y$ is an element $x \in X$, such that $f(x) = y$.

Remark: We can always modify a function $f_1: X \to Y$ so it is surjective by replacing the codomain Y with the image of f, i.e. now $f_2: X \to Im(f)$, with $f_2(x) = f_1(x)$ $\forall x \in X$.

Example: Determine if the function
$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 2x+3$$
is an injection, bijection, surjection.

Solution: First let us determine if f is injective.

Consider $x_1, x_2 \in \mathbb{R}$, with $f(x_1) = f(x_2)$.
Thus $(2x_1 + 3 = 2x_2 + 3) \Leftrightarrow (2x_1 = 2x_2) \Leftrightarrow (x_1 = x_2)$.
So we showed $(f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$, so f is injective.

Second, let's determine if f is surjective.

Consider any $y \in \mathbb{R}$, we'd like to determine if $\exists x \in \mathbb{R}$ with $f(x) = y$.

But $2x + 3 = y \in \mathbb{R} \Leftrightarrow x = \frac{y-3}{2} \in \mathbb{R}$, so $\forall y \in \mathbb{R}$, $\exists x \in \mathbb{R}$, $2x+3=y$. Thus f is surjective.

As f is both injective and surjective, it is a bijection.

More efficient solution

Let y be an element of the co-domain $\mathbb{R}$.

Then,

$$x \text{ is a pre-image of } y$$

$$\iff y = f(x) = 2x+3$$

$$\iff x = \frac{y-3}{2}.$$

This shows each $y \in Y$ has exactly one pre-image in $\mathbb{R}$ $x = \frac{y-3}{2}$, so f is a bijection.

Exercise: Consider $f : \mathbb{R} \to \mathbb{R}$
$$x \mapsto x^2.$$
Determine if f is one-to-one and if f is onto.

Repeat this exercise with

(1) $f : \mathbb{R} \to \mathbb{R}^{\geq}$
$$x \to x^2$$

(2) $f : \mathbb{R} \to \mathbb{R}$
$$x \mapsto x^3 + 1.$$

Example: $f : \mathbb{R} \to \mathbb{R}$
$$x \mapsto 4x^2 - 4x + 2$$
Determine if f is one-to-one and if f is onto.

Solution: (1) Injectivity:
Let $x_1, x_2 \in \mathbb{R}$ be such that $f(x_1) = f(x_2)$. That is

$$(4x_1^2 - 4x_1 + 2 = 4x_2^2 - 4x_2 + 2) \iff (x_1^2 - x_1 = x_2^2 - x_2)$$
$$\iff (x_1 - x_2)(x_1 + x_2) - (x_1 - x_2) = 0 \iff (x_1 - x_2)(x_1 + x_2 - 1) = 0$$
$$\Rightarrow (x_1 = x_2) \vee (x_1 = -x_2 + 1).$$
So f is not injective because $f(1) = f(0) = 2$ and $1 \neq 0$.

(2) Surjectivity:

Notice that $4x^2-4x+2 = 4x^2-4x+1+1$
$$= (2x-1)^2+1$$
$$\geqslant 1.$$

Thus $y=0$ does not have a pre-image. So f is not surjective.

Bijections and inverses

Definition: A function $f: X \to Y$ is invertible if there is a function $g: Y \to X$ such that

$$f(x)=y \Leftrightarrow x = g(y) \quad \forall x \in X, y \in Y.$$

g is called the inverse of f.

Example: The function $f: \mathbb{R} \to \mathbb{R}$, given by $f(x) = 2x+3$ is invertible because $2x+3=y \Leftrightarrow x = \frac{y-3}{2}$ $\forall x, y \in \mathbb{R}$. Here, $g(y) = \frac{y-3}{2}$.

Theorem: A function $f: X \to Y$ is invertible if and only if it is a bijection. Moreover, if f is invertible, its inverse function is unique.

Proof: (I) Proof that 'f is invertible $\Rightarrow$ f is bijective'.

Suppose f is invertible, then there is a function $g: Y \to X$ s.t. $y = f(x) \Leftrightarrow x = g(y)$ $\forall x \in X, y \in Y$.

(a) We will now show that f is injective. To that end, let x_1, x_2 be such that $f(x_1) = f(x_2)$.

Then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$, so $x_1 = x_2$ and f is injective.

(b) Now, we will show that f is surjective. Let y be any element of Y, and set $x = g(y)$. Then $f(x) = y$. So we have shown that $\forall y \in Y, \exists x \in X, f(x) = y$, i.e, surjectivity.

By (a) and (b), f is bijective.

(II) Proof that 'f is bijective $\Rightarrow$ f is invertible'

Suppose f is bijective and let $y \in Y$ be any element of Y. By bijectivity of f, there exists one unique element $x \in X$ such that

$f(x) = y$. Now, we simply define a function $g: Y \to X$ via $(g(y) = x$ where $x = f(y))$. Note that since x is unique, g is a function.

So now, by construction, we have

$$y = f(x) \iff g(y) = f(x),$$

so f is invertible and g is its inverse.

(III) Proof that 'f is invertible $\Rightarrow$ its inverse is unique'

Suppose there are two functions $g_1, g_2 : Y \to X$ that are both inverses of $f: X \to Y$.
For g_1 to be different from g_2 there

must be at least one element $y \in Y$ s.t. $g_1(y) \neq g_2(y)$.

Let $x_1 = g_1(y)$ and $x_2 = g_2(y)$,

then $\quad x_1 = g_1(y) \iff f(x_1) = y$

and $\quad x_2 = g_2(y) \iff f(x_2) = y$

so that $f(x_1) = f(x_2)$. But f is a bijection, hence one-to-one, so $f(x_1) = f(x_2)$ $\Rightarrow x_1 = x_2 \Rightarrow g_1(y) = g_2(y)$ showing that $g_1 = g_2$.

$\blacksquare$

Proposition: $f: X \to Y$ and $g: Y \to X$ are inverses of each other if and only if

$$g \circ f = I_X \text{ and } f \circ g = I_Y.$$

Proof: If f and g are inverses of each other, then

$$y = f(x) \iff x = g(x) \quad \forall x \in X,\ \forall y \in Y$$

For any $x \in X$, let $y = f(x)$, then $g(y) = g(f(x)) = x$

i.e. $g \circ f = I_X$

Similarly, for any $y \in Y$, let $x = g(y)$ so

$$f(x) = f(g(y)) = y$$

so $f \circ g = I_Y$. ∎

Functions on sets:

Let $f : X \to Y$, we may define a new function

$$\vec{f} : P(X) \longrightarrow P(Y)$$
$$\vec{f}(A) = \{ f(x) \mid x \in A \}.$$

We can also define a function $\overleftarrow{f}$

$$\overleftarrow{f} : P(Y) \to P(X)$$
$$\overleftarrow{f}(B) = \{ x \in X \mid f(x) \in B \}$$

Most mathematicians and textbooks, without risk of confusion drop the arrow on $\vec{F}$ and use F instead.

They also use F^{-1} in place of $\overleftarrow{F}$.